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# A SHORT PROOF OF DE SHALIT'S CUP PRODUCT FORMULA

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Abstract. We give a short proof of a formula of de Shalit, expressing the cup product of two vector valued one forms of the second kind on a Mumford curve in terms of Coleman integrals and residues. The proof uses the notion of double indices on curves and their reciprocity laws.

## 1. Introduction

In [dS88] de Shalit proved a formula for the cup product of two vector valued differential forms on a Mumford curve. This is based on an earlier partial result of his [dS89] for two holomorphic differentials. This formula was later reproved by Iovita and Spiess [IS03]. The goal of this short note is to give an alternative short proof of de Shalit's formula, based on the theory of the double index [Bes00, Section 4].

Let us state de Shalit's result. Let $K$ be a finite extension of $\mathbb{Q}_p$. Consider a Mumford curve $\mathcal{H}/\Gamma$, where $\Gamma \subset \mathrm{PGL}_2(K)$ is a Schottky group and $\mathcal{H} \subset \mathbb{P}^1_K$ is the rigid analytic space obtained by removing the limit points of $\Gamma$. Let $V$ be a finite dimensional $K$-vector space with a representation of $\Gamma$. The group $\Gamma$ acts on the space of $V$-valued differential forms on $\mathcal{H}$, $\Omega^1(\mathcal{H}, V)$, by the rule

$$\gamma(\sum \omega_i v_i) = \sum (\gamma^{-1})^* \omega_i \gamma(v_i)$$

(compare [dS88, 1.1]). We let it act by the same formula on spaces of functions. A $V$-valued differential one-form $\omega$ on $\mathcal{H}$ with values in $V$ is $\Gamma$-invariant if $\gamma(\omega) = \omega$ for every $\gamma \in \Gamma$. It is of the second kind if its residues (with values in $V$, computed coordinatewise, in any basis), are 0 at any point $z \in \mathcal{H}$. Let $\langle\ \rangle$ be a $\Gamma$-invariant bilinear form on $V$. The cup product of two $\Gamma$-invariant $V$-valued one forms of the second kind $\omega$ and $\eta$ can be described by the formula

$$\omega \cup \eta = \sum_{z \in \Gamma \backslash \mathcal{H}} \mathrm{Res}_z \langle F_\omega, \eta \rangle ,$$

where $F_\omega$ is any primitive of $\omega$ locally near $z$, which exists (formally) because of the residue of $\omega$ at $z$ is 0, and is independent of the choice of the primitive because the residue of $\eta$ at $z$ is 0. Note that the expression to be summed indeed depends only on $z$ modulo $\Gamma$.

An open annulus is a rigid space isomorphic to the space $s < |z| < r$. An orientation on an annulus may be described as a choice of a parameter $z$ as above, with two parameters considered equivalent if they give the residue, as defined below. An annulus together with an orientation is called an oriented annulus. A differential form $\omega$ on an oriented annulus $e$ has a residue $\mathrm{Res}_e\, \omega$ such that $\mathrm{Res} \sum a_i z^i dz = a_{-1}$.

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It can be shown that there are only two orientations, giving residues differing by multiplication by $-1$. By choosing a basis for $V$ the residue extends easily to $V$ valued differential forms.

We now recall [dS89, Definition 2.5] that the action of $\Gamma$ on $\mathcal{H}$ has a good fundamental domain in the following sense: There are pairwise disjoint closed $K$-rational discs $B_i$ and $C_i$, $i = 1, \dots, g$ and open annuli $b_i$, $c_i$, and elements $\gamma_i \in \Gamma$, such that the following holds:

(1) The $\gamma_i$ freely generate $\Gamma$.
(2) The unions $B_i \cup b_i$ and $C_i \cup c_i$ are open discs, still pairwise disjoint.
(3) For each $i$, $\gamma_i$ maps $B_i$ isomorphically onto the complement of $C_i \cup c_i$ and $b_i$ isomorphically onto $c_i$.
(4) The complement of $\bigcup_i (B_i \cup b_i \cup C_i)$ is a fundamental domain for $\Gamma$.

We give the annuli $c_i$ and $b_i$ the orientation given by the discs $C_i$ and $B_i$ respectively, i.e., one given by parameters extending to $C_i \cup c_i$ and taking the value 0 on $C_i$ (respectively with $b_i$ and $B_i$). Thus, $c_i$ is oriented in the same way as in [dS88, 1.5] while $b_i$ is oriented in the reversed direction to loc. cit. (the $b_i$'s do not show up in the formula). With this choice, $\gamma_i : b_i \to c_i$ is orientation reversing.

de Shalit's formula involves Coleman integration of holomorphic $V$-valued 1 forms. While this can be described in a completely elementary way since we are dealing with subdomains of the projective line [GvdP80, P. 41], we will use the more involved theory of Coleman [CdS88] and adapt it to our case by choosing a basis of $V$ and then integrate coordinate by coordinate. This is clearly independent of the choice of a basis because Coleman integration is linear (up to constant). The key property of Coleman integration is its functoriality. It immediately implies that from the property $\gamma\omega = \omega$ we may deduce that for any $\gamma \in \Gamma$ the function $\gamma(F_\omega) - F_\omega$ is constant. We can now state the main theorem.

**Theorem 1.1** ([dS88, Theorem 1.6]). *With the data above we have*

$$\omega \cup \eta = \sum_i \langle \gamma_i F_\omega - F_\omega, \operatorname{Res}_{c_i} \eta \rangle - \langle \operatorname{Res}_{c_i} \omega, \gamma_i F_\eta - F_\eta \rangle \ .$$

The main ingredient in the present proof is the theory of double indices and their reciprocity laws on curves [Bes00, Section 4]. We need a very easy extension of this theory to vector valued differential forms. Once this has been described, the proof is an easy computation.

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## 2. Double indices of vector valued differential forms

In this section we describe a rather straightforward generalization of the theory of double indices [Bes00, Section 4] to the case of vector valued one forms. The extension is fairly trivial since we consider only constant coefficients. We work over $\mathbb{C}_p$ for convenience.

Let $A$ be either the field of meromorphic functions in the variable $z$ over $\mathbb{C}_p$ or the ring of rigid analytic functions on an annulus $\{r < |z| < s\}$ over $\mathbb{C}_p$. Let $A_{\log} := A[\log(z)]$ and let $A_{\log,1} \subset A_{\log}$ be the subspace of $F \in A_{\log}$ which are linear in $\log(z)$, a condition which is equivalent to $dF \in A dz$.

**Definition 2.1.** [Bes00, Proposition 4.5] The double index, $\operatorname{ind}(\ ) : A_{\log,1} \times A_{\log,1} \to \mathbb{C}_p$ is the unique antisymmetric bilinear pairing such that $\operatorname{ind}(F, G) = \operatorname{Res} F dG$, whenever $F \in A$.

Suppose now that $C$ is a proper smooth curve over $\mathbb{C}_p$ with good reduction, and that $U$ is a rigid analytic space obtained from $C$ by removing discs $D_i$ of the form $|z_i| \le r$, with $r < 1$, where the reduction of $z_i$ is a local parameter near a point $x_i$ of the reduction. Let us call these domains *simple domains*. To the disc $D_i$ corresponds the annulus $e_i$ given by the equation $r < |z_i| < 1$, which is contained in $U$ and oriented by $z_i$.

Choose a branch of the $p$-adic logarithm. Given a rigid one form $\omega \in \Omega^1(U)$, Coleman's theory provides us with a unique up to constant, locally analytic function $F_\omega$ on $U$ with the property that $dF_\omega = \omega$. Restricted to the annuli $e_i$ these clearly belong to $A_{\log,1}$ and one can therefore define, for two such functions $F_\omega$ and $F_\eta$ the double index $\operatorname{ind}_{e_i}(F_\omega, F_\eta)$. It follows from [Bes00, Lemma 4.6] that this index depends only on the orientation. One of the main technical results of [Bes00] is the following.

**Proposition 2.2** ([Bes00, Proposition 4.10])**.** *We have* $\sum_i \operatorname{ind}_{e_i}(F_\omega, F_\eta) = \Psi(\omega) \cup \Psi(\eta)$, *where* $\Psi : H^1_{\mathrm{dR}}(U) \to H^1_{\mathrm{dR}}(C)$ *is a certain projection.*

We will only need the following immediate Corollary, which follows because $H^1_{\mathrm{dR}}(\mathbb{P}^1/\mathbb{C}_p) = 0$.

**Corollary 2.3.** *Suppose that* $C = \mathbb{P}^1$. *Then, in the situation above,* $\sum_i \operatorname{ind}_{e_i}(F_\omega, F_\eta) = 0$.

We can now extend the theory to vector valued differential forms in a rather trivial way. Suppose we are given a finite dimensional $\mathbb{C}_p$-vector space with a bilinear form $\langle , \rangle$.

**Definition 2.4.** Chose bases $\{v_i\}$ and $\{u_i\}$ for $V$. Suppose that the $V$-valued Coleman functions $F_\omega$ and $F_\eta$ are written as

$$F_\omega = \sum F_{\omega_i} v_i \ ,$$
$$F_\eta = \sum F_{\eta_i} v_i \ .$$

Then, the local index $\operatorname{ind}_e(F_\omega, F_\eta)$ is given by

$$\operatorname{ind}_e(F_\omega, F_\eta) = \sum_{i,j} \operatorname{ind}_e(F_{\omega_i}, F_{\eta_j}) \langle v_i, u_j \rangle \ .$$

It is easy to check that this definition does not depend on the choice of bases. An easy consequence of the definitions is the following.

**Proposition 2.5.** *Suppose that* $\operatorname{Res}_e \omega = 0$. *Then* $\operatorname{ind}_e(F_\omega, F_\eta) = \operatorname{Res}_e \langle F_\omega, \eta \rangle$ *while* $\operatorname{ind}_e(F_\eta, F_\omega) = -\operatorname{Res}_e \langle \eta, F_\omega \rangle$.

We now restrict to the case $C = \mathbb{P}^1$ but consider more general subdomains $U$, obtained by removing closed discs $D_i = |z - \alpha_i| = r_i$, including the case of removing a point when $r_i = 0$. For each $i$ we consider an annulus $e_i$ in $U$ surrounding $D_i$, in such a way that the open discs $D_i \cup e_i$ are still disjoint We will call the $e_i$ the *annuli ends* of $U$. It is easy to see that $U$ can be obtained by gluing simple domains $U' \in \mathbb{P}^1$ along annuli. Note that the $U'$'s are glued along annuli with reversed orientations.

Given $\omega \in \Omega^1(U,V)$ one can define its Coleman integral $F_\omega$ first on each of the $U'$'s as before and then by adjusting constants along the annuli. The intersection graph of the $U'$'s is a tree so there is always a way of choosing an integral globally. This construction coincides with the definition of Coleman integrals in [GvdP80].

**Proposition 2.6.** *In the situation described above we have, for any rigid $V$-valued one-form on $U$, $\sum_i \operatorname{ind}_{e_i}(F_\omega, F_\eta) = 0$.*

*Proof.* The case $V$ trivial and $U$ simple is Corollary 2.3. We next consider the case $U = U'_1 \cup U'_2$ with $U'_1$ and $U'_2$ glued along an annulus $e$. Since $e$ has reversed orientations when considered in $U'_1$ and $U'_2$, the double index $\operatorname{ind}_e$ has a reverse sign in these two cases by [Bes00, Lemma 4.6]. Thus, the result for $U$ follows from those for $U'_1$ and $U'_2$. Now, the case of a general $U$, still with trivial $V$, follows immediately. The general case follows by choosing bases. □

**Proposition 2.7.** *Let $e$ be an annulus in $\mathcal{H}$ and let $\gamma \in \Gamma$. For $\omega \in \Omega^1(e,V)$ let $F_\omega$ be its integral. Then $\gamma F_\omega$ is a Coleman integral of $\gamma(\omega)$ on $\gamma(e)$, furthermore, If $\eta$ is another such form, then we have*

$$\operatorname{ind}_e(F_\omega, F_\eta) = \pm \operatorname{ind}_{\gamma(e)}(\gamma(F_\omega), \gamma(F_\eta)) ,$$

*depending on whether $\gamma$ is orientation reversing or saving.*

*Proof.* We choose a basis $\{v_i\}$ of $V$ and we let $\{u_i\}$ be the dual basis with respect to $\langle , \rangle$. Then, since $\langle , \rangle$ is $\Gamma$-invariant, the bases $\{\gamma(v_i)\}$ and $\{\gamma(u_i)\}$ are also dual to each other. This implies that if $F_\omega = \sum f_i v_i$ while $F_\eta = \sum g_i u_i$, then

$$\operatorname{ind}_e(F_\omega, F_\eta) = \sum_i \operatorname{ind}_e(f_i, g_i)$$

and

$$\operatorname{ind}_{\gamma(e)}(\gamma(F_\omega), \gamma(F_\eta)) = \sum_i \operatorname{ind}_{\gamma(e)}((\gamma^{-1})^* f_i, (\gamma^{-1})^* g_i) .$$

But by [Bes00, Lemma 4.6] we have, for each $i$,

$$\operatorname{ind}_e(f_i, g_i) = \pm \operatorname{ind}_{\gamma(e)}((\gamma^{-1})^* f_i, (\gamma^{-1})^* g_i) ,$$

depending on whether $\gamma^{-1}$ is orientation reversing or preserving, and the result follows immediately from this. □

## 3. The proof

*Proof of Theorem 1.1.* By the remark following Equation (5) in [dS88] we may assume that $b_i$ and $c_i$ contain no poles of $\omega$ and $\eta$. Consider the domain $\mathcal{F} = \mathbb{P}^1 - \bigcup_i (B_i \cup C_i)$, which is of the type considered in Section 2, and its annuli ends are the $b_i$ and $c_i$. It follows from the description of the fundamental domain for $\Gamma$ that $\mathcal{F} - \bigcup_i (c_i \cup b_i)$ contains exactly one out of every $\Gamma$ class of every singularity of either forms. It follows that

$$\omega \cup \eta = \sum_{x \in \mathcal{F}} \operatorname{Res}_x \langle F_\omega, \eta \rangle = \sum_{x \in \mathcal{F}} \operatorname{ind}_x(F_\omega, F_\eta) = -\sum_i (\operatorname{ind}_{b_i}(F_\omega, F_\eta) + \operatorname{ind}_{c_i}(F_\omega, F_\eta))$$

where the last equality follows from Proposition 2.6. We now observe that since $\gamma_i$ is orientation reversing we have by Proposition 2.7 that $\operatorname{ind}_{b_i}(F_\omega, F_\eta) = -\operatorname{ind}_{c_i}(\gamma_i F_\omega, \gamma_i F_\eta)$.

Therefore

$$
\begin{aligned}
&-\left(\operatorname{ind}_{b_i}(F_\omega, F_\eta) + \operatorname{ind}_{c_i}(F_\omega, F_\eta)\right)\\
&= \operatorname{ind}_{c_i}(\gamma_i F_\omega, \gamma_i F_\eta) - \operatorname{ind}_{c_i}(F_\omega, F_\eta)\\
&= \operatorname{ind}_{c_i}(\gamma_i F_\omega - F_\omega, \gamma_i F_\eta) + \operatorname{ind}_{c_i}(F_\omega, \gamma_i F_\eta - F_\eta)\\
&= \operatorname{Res}_{c_i}\langle \gamma_i F_\omega - F_\omega, \gamma_i \eta\rangle - \operatorname{Res}_{c_i}\langle \omega, \gamma_i F_\eta - F_\eta\rangle \quad \text{by Proposition 2.5}\\
&= \langle \gamma_i F_\omega - F_\omega, \operatorname{Res}_{c_i}\eta\rangle - \langle \operatorname{Res}_{c_i}\omega, \gamma_i F_\eta - F_\eta\rangle\ .
\end{aligned}
$$

The theorem follows immediately. □

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